Yang-Lee edge singularity of a one-dimensional Ising ferromagnet with arbitrary spin

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(Received 25 March 1998; revised manuscript received 20 May 1998)

It is shown that for a one-dimensional lattice system in a purely imaginary magnetic field, if the interaction is finite range, the nature of the Yang-Lee edge singularity is universal, independent of the spin and interaction strengths. The edge singularity corresponds to the twofold degeneracy of the largest eigenvalues of the transfer matrix. For the Blume-Emery-Griffiths ferromagnet, the tricritical point and the edge pseudosingularity may exist. The tricritical point corresponds to the triple degeneracy of the eigenvalues. The edge pseudosingularity corresponds to the twofold degeneracy of the nonlargest eigenvalues. [S1063-651X(98)11909-8]

PACS number(s): 05.50.+q, 75.40.Cx, 75.10.Hk, 64.60.Cn

I. INTRODUCTION

In 1952, Yang and Lee [1] opened a new way to study phase transition. They called attention to the zeros of the grand partition function in the complex fugacity plane. They showed that in the thermodynamic limit the zero distribution approaches the positive real axis and gives the transition point. In application to the ferromagnetic Ising model, they considered the zeros of the partition function in the complex magnetic plane and proved the famous circle theorem. The Yang-Lee circle theorem states that the zeros of the partition function in the complex magnetic plane are distributed on a unit circle. Later this theorem was extended to many ferromagnetic systems, such as the higher-spin Ising model [2], Ising models with multiple spin interactions, the quantum Heisenberg model [3], the classical XY and Heisenberg model [4], and some continuous spin systems [5].

Above the zero-field critical temperature, the zeros do not come close to the real h axis in the thermodynamic limit and the free energy is not analytic in h. There exists a gap in the imaginary h axis, where zeros are void. Since the gap size depends on the temperature, one can envision a critical line $h=ih_0(T_c)$ (here h_0 is real) along which the free energy becomes singular, $F\sim(h-ih_0)^\theta$ (here θ is a critical exponent) [6–8]. This singularity was termed the Yang-Lee edge singularity by Fisher. Fisher [9] proved that the edge singularities $h=ih_0$, representing zeros lying closest to real values of the field, are closely analogous to the conventional critical point and that the relevant scaling laws are applicable. Furthermore, the universality should hold for them too and the critical exponents of these singularities are independent of the detailed lattice structure and interaction strengths, and depend only on the spatial dimensions and symmetry property of order parameter.

Since the edge singularity has the most important influence on the equation of state of a ferromagnet, there have been many studies about it. These include the edge singularity in the Ising model [10], in the hierarchical model [11], in the spherical model [12], in the classical n-vector models and the quantum Heisenberg model [13], as well as in the relation with conformal invariance in two dimensions [14] and in the relation with the critical behavior of branched polymers [15], etc.

This paper is organized as follows. In Sec. II, it is shown that phase transition is marked by the occurrence of the twofold degeneracy of the largest eigenvalue of the transfer matrix. For a one-dimensional (1D) lattice system in a real magnetic field, if the interaction is finite range, no phase transition occurs at a finite temperature. For the same system in a purely imaginary magnetic field, the nature of the Yang-Lee edge singularity is universal. In Sec. III the Yang-Lee edge singularity of 1D Ising ferromagnets with spin 1/2, 1, and 3/2 is studied, respectively. In Sec. IV, the Yang-Lee edge singularity of the 1D spin-1 Blume-Emery-Griffiths model is studied. In Sec. V, a summary of this paper is given.

II. GENERAL CONDITION OF PHASE TRANSITION

We consider a lattice system in a real magnetic field. We assume that its transfer matrix is given by an $L\times L$ matrix (L is finite), whose eigenvalues are obtained from the characteristic equation

$$\lambda^L+a_1\lambda^{L-1}+a_2\lambda^{L-2}+\cdots+a_{L-1}\lambda+a_L=0, \tag{1}$$

where a_l are coefficients determined by the model. Let λ_m be the largest eigenvalue and N be the total number of the lattice points. In the thermodynamic limit, the partition function, the free energy, the magnetization, and the susceptibility are given, respectively, by

$$Z=\lambda_m^N, \tag{2}$$

$$f=F/N=-kT\ \ln\lambda_m, \tag{3}$$

$$M=-\left(\frac{\partial f}{\partial h}\right)_T=\beta^{-1}\frac{1}{\lambda_m}\frac{\partial\lambda_m}{\partial h}, \tag{4}$$

$$\left(\frac{\partial M}{\partial h}\right)_T=\beta^{-1}\left[\lambda_m^{-1}\frac{\partial^2\lambda_m}{\partial h^2}-\lambda_m^{-2}\left(\frac{\partial\lambda_m}{\partial h}\right)^2\right]. \tag{5}$$

From Eq. (5), we deduce that the condition of phase transition $(\partial M/\partial h)_T\rightarrow\infty$ implies that $(\partial\lambda_m/\partial h)$ or $(\partial^2\lambda_m/\partial h^2)$ become infinity. In order to find the partial derivatives of λ_m with respect to h, we differentiate Eq. (1) with respect to h,

$$\frac{\partial \lambda_m}{\partial h}\sum_{l=0}^{L-1}(L-l)a_l\lambda_m^{L-l-1}+\sum_{l=0}^{L}\frac{\partial a_l}{\partial h}\lambda_m^{L-l}=0, \tag{6}$$

$$\frac{\partial^2 \lambda_m}{\partial h^2}\sum_{l=0}^{L-1}(L-l)a_l\lambda_m^{L-l-1} +\left(\frac{\partial \lambda_m}{\partial h}\right)^2\sum_{l=0}^{L-2}(L-l)(L-l-1)a_l\lambda_m^{L-l-2} +2\frac{\partial \lambda_m}{\partial h}\sum_{l=0}^{L-1}(L-l)\frac{\partial a_l}{\partial h}\lambda_m^{L-l-1}+\sum_{l=0}^{L}\frac{\partial^2 a_l}{\partial h^2}\lambda_m^{L-l}=0, \tag{7}$$

where $a_0=1$. Since in general a_l and their derivatives with respect to h do not diverge, the condition of phase transition requires that

$$\sum_{l=0}^{L-1}(L-l)a_l\lambda_m^{L-l-1}=0. \tag{8}$$

The condition of phase transition is determined by Eqs. (1) and (8). Equation (1) is an algebraic equation of order L. Equation (8) acts as a constraint equation to be satisfied by the largest eigenvalue at the critical point. Thus Eq. (1) has $(L-1)$ distinct roots. Therefore, the largest eigenvalue is twofold degenerate. We obtain an interesting conclusion that *for a lattice system in a real magnetic field, the condition of phase transition requires that the largest eigenvalue of the transfer matrix be twofold degenerate*. In his solution of the square lattice spin-1/2 Ising model, Onsager [16] noted that below the critical point, the largest eigenvalue is degenerate (twofold degeneracy). The occurrence of the twofold degeneracy of the largest eigenvalue is a sufficient condition of phase transition [17,18].

Let us expand the largest eigenvalue around the critical point, $\lambda_m=\lambda_m^0+\delta\lambda$, and determine the critical behavior. Here λ_m^0 is the largest eigenvalue at the critical point and satisfies Eqs. (1) and (8). Substituting λ_m into Eq. (1) and expanding, we get

$$\sum_{l=0}^{L}a_l(\lambda_m^0+\delta\lambda)^{L-l}=b_2(\delta\lambda)^2+b_3(\delta\lambda)^3+b_4(\delta\lambda)^4+\cdots =0, \tag{9}$$

where

$$b_n=\sum_{l=0}^{L-n}C_{L-l}^{n}a_l(\lambda_m^0)^{L-l-n}. \tag{10}$$

Keeping the larger terms, we get

$$b_2+b_3\delta\lambda+b_4(\delta\lambda)^2=0 \tag{11}$$

or

$$\lambda_m=\lambda_m^0+\frac{-b_3\pm\sqrt{b_3^2-4b_2b_4}}{2b_4}. \tag{12}$$

From Eqs. (4) and (5) we obtain the singular parts of M and $(\partial M/\partial h)_T$,

$$M\sim(b_3^2-4b_2b_4)^{-1/2} \tag{13}$$

and

$$\left(\frac{\partial M}{\partial h}\right)_T\sim(b_3^2-4b_2b_4)^{-3/2}. \tag{14}$$

Thus we see that the phase transition condition (twofold degeneracy) requires $b_3^2-4b_2b_4=0$, which implies $M\to\infty$. On the other hand, when the magnetic field is real, M should be finite. Therefore we find that *for any lattice system in a real magnetic field, if its transfer matrix is finite dimensional, no phase transition occurs at a finite temperature*.

If L were infinite, we could not have an expansion as Eq. (9) and the origin of singularity would be quite different. Thus the above conclusion will not be valid in general.

For a 1D lattice system in a real magnetic field, if the interaction is finite range, the transfer matrix is finite dimensional, independent of lattice size. So no phase transition occurs at a finite temperature. For a higher-dimensional lattice system in a real magnetic field, if the lattice size is finite, then its transfer matrix is finite dimensional and no phase transition occurs. These are well-known facts [19,20].

For a 1D lattice system, let us turn our attention to the more interesting case in which the magnetic field is purely imaginary. If all eigenvalues are real, from Eq. (2), we know $Z\to\lambda_m^N>0$ in the thermodynamic limit and no Yang-Lee zeros appear. Therefore, we deduce that in the gap where Yang-Lee zeros are absent, the eigenvalues of the transfer matrix must be real. We can use the general condition of phase transition, Eqs. (1) and (8), to get the Yang-Lee edge singularity.

We see that the singularity stems from the square root in Eq. (12). This means that the critical exponents are universal. Thus we conclude that *for a 1D lattice system in a purely imaginary magnetic field, if its transfer matrix is finite dimensional and the Yang-Lee edge singularity exists, then the nature of edge singularity is universal, independent of spin and interaction strengths*. We will confirm this in later sections.

III. 1D SPIN-*S* PURE ISING FERROMAGNET

The partition function of a 1D spin-S Ising model in the presence of a magnetic field h is given by

$$Z=\sum_{\{S_i\}}\exp\left(\beta\sum_{j=1}^{N}JS_jS_{j+1}+\beta h\sum_{j=1}^{N}S_i\right), \tag{15}$$

where $S_i=-S,-S+1,\ldots,S-1,S$ and $J>0$. The periodic boundary condition is imposed so that $S_{N+1}=S_1$. The transfer matrix is given by

$$\langle S_j|V|S_{j+1}\rangle=\exp[\beta JS_jS_{j+1}+\beta h(S_j+S_{j+1})/2]. \tag{16}$$

The transfer matrix is an $L\times L$ matrix with $L=2S+1$. Then the partition function can be expressed as

$$Z=\sum_{S_j}\langle S_1|V|S_2\rangle\langle S_2|V|S_3\rangle\cdots\langle S_N|V|S_1\rangle$$
$$=\mathrm{Tr}\ V^N=\lambda_1^N+\cdots+\lambda_L^N, \tag{17}$$

where $\lambda_1,\ldots,\lambda_L$ are the eigenvalues of the transfer matrix, given by

$$\lambda^L+a_1\lambda^{L-1}+a_2\lambda^{L-2}+\cdots+a_{L-1}\lambda+a_L=0. \tag{18}$$

Because of symmetry, $a_l(-h,T)=a_l(h,T)$. We will consider the case when h is purely imaginary, $h=ih_I$. In this case a_l will be all real.

A. $S=1/2$

The transfer matrix is

$$V=\begin{pmatrix} e^{\beta(J+2h)/4} & e^{-\beta J/4} \\ e^{-\beta J/4} & e^{\beta(J-2h)/4}\end{pmatrix}.$$

The eigenvalues can be determined easily,

$$\lambda_{1,2}=e^{\beta J/4}[\cosh\ \beta h/2\pm(\sinh^2\beta h/2+e^{-\beta J})^{1/2}]. \tag{19}$$

We consider the case in which h is purely imaginary. The two eigenvalues are real or complex conjugate depending on the value of $(\sinh^2\beta h/2+e^{-\beta J})$. If $(\sinh^2\beta h/2+e^{-\beta J})\geqslant 0$, the two eigenvalues are real and $\lambda_1\geqslant\lambda_2$. In the thermodynamic limit, we obtain the free energy,

$$f=-J/4-\beta^{-1}\ln[\cosh\ \beta h/2+(\sinh^2\beta h/2+e^{-\beta J})^{1/2}], \tag{20}$$

and the magnetization,

$$M=\frac{\sinh\ \beta h/2}{(\sinh^2\beta h/2+e^{-\beta J})^{1/2}}. \tag{21}$$

Thus we have

$$\left(\frac{\partial M}{\partial h}\right)_T=\frac{\beta\cosh\ \beta h/2}{(\sinh^2\beta h/2+e^{-\beta J})^{1/2}}-\frac{\beta\sinh\ \beta h/2\cosh\ \beta h/2}{(\sinh^2\beta h/2+e^{-\beta J})^{3/2}}. \tag{22}$$

The phase transition condition, $(\partial h/\partial M)_T=0$, requires that $\sinh^2\beta h/2+e^{-\beta J}=0$. Let $h=ih_0$. The critical line is given by

$$\sin\ \beta_c h_0/2=e^{-\beta_c J/2}. \tag{23}$$

From above we find that the occurrence of the Yang-Lee edge singularity corresponds to the twofold degeneracy of eigenvalues of the transfer matrix.

If $\sinh^2\beta h/2+e^{-\beta J}<0$, the two eigenvalues are complex conjugate

$$\lambda_{1,2}=e^{\beta J/4}[\cosh\ \beta h/2\pm i|\sinh^2\beta h/2+e^{-\beta J}|^{1/2}]. \tag{24}$$

On the other hand, the Yang-Lee zeros are given by $Z=0$, i.e.,

$$\left(\sinh^2\frac{\beta h}{2}+e^{-\beta J}\right)^{1/2}=i\cosh\frac{\beta h}{2}\tan\pi\frac{2n+1}{2N}$$
$$(n=0,1,\ldots,N-1). \tag{25}$$

We see that if $\sinh^2\beta h/2+e^{-\beta J}<0$, Yang-Lee zeros exist and are given by

$$\sin^2\frac{\beta h_I}{2}=\sin^2\pi\frac{2n+1}{2N}+e^{-\beta J/2}\cos^2\ \pi\frac{2n+1}{2N}. \tag{26}$$

Therefore, we find that in the gap the eigenvalues are real and no Yang-Lee zeros appear. At the edge $h_I=\pm h_0(T_c)$, the eigenvalues become twofold degenerate precisely. On the rest of the imaginary magnetic field axis, the complex conjugate eigenvalues appear and also Yang-Lee zeros appear.

Let us define the critical exponents near the critical line in the ordinary sense: (i) $M(h=ih_0)\to(T-T_c)^\beta$; (ii) $M(T=T_c)\to(h-ih_0)^{1/\delta}$; (iii) the specific heat $C(h=ih_0)\to(T-T_c)^{-\alpha}$; (iv) $\chi=(\partial M/\partial h)(h=ih_0)\to(T-T_c)^{-\gamma}$; (v) the correlation length $\xi(h=ih_0)\to(T-T_c)^{-\nu}$; (vi) the spin-spin correlation function $g(r)\to r^{-d+2-\eta}$.

From Eqs. (20)–(22) we find that $\beta=-1/2$, $\delta=-2$, $\alpha=3/2$, and $\gamma=3/2$. Other exponents can be obtained from the spin-spin correlation function that has been calculated in [21,22]. The result is

$$g(r)=(1-M^2)e^{-r/\xi}, \tag{27}$$

where

$$\xi=[\ln(\lambda_1/\lambda_2)]^{-1}. \tag{28}$$

Using Eq. (19) we find that $\xi(h=ih_0)\to(T-T_c)^{-1/2}$ as $T\to T_c$ and $\nu=1/2$. From Eq. (27), we see that η is not definite. If we take $\eta=-1$, the critical exponents satisfy all scaling relations: $\alpha+2\beta+\gamma=2$, $\beta(\delta-1)=\gamma$, $\gamma=\nu(2-\eta)$, $\delta=(d+2-\eta)/(d-2+\eta)$, and $\alpha=2-\nu d$.

B. $S=1$

The transfer matrix is given by

$$V=\begin{pmatrix} e^{\beta(J+h)} & e^{\beta h/2} & e^{-\beta J} \\ e^{\beta h/2} & 1 & e^{-\beta h/2} \\ e^{-\beta J} & e^{-\beta h/2} & e^{\beta(J-h)}\end{pmatrix}.$$

The eigenvalues of the transfer matrix are obtained from

$$\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0, \tag{29}$$

where

$$a_1=-[1+e^{\beta J}(e^{\beta h}+e^{-\beta h})], \tag{30}$$

$$a_2=(e^{\beta J}-1)(e^{\beta h}+e^{-\beta h})+e^{2\beta J}-e^{-2\beta J}, \tag{31}$$

$$a_3=e^{-2\beta J}-e^{2\beta J}+2(e^{\beta J}-e^{-\beta J}). \tag{32}$$

We consider the case in which h is purely imaginary. All of a_1, a_2, and a_3 are real. Define $Q=(3a_2-a_1^2)/9$ and $R=(9a_1a_2-27a_3-2a_1^3)/54$. $D=Q^3+R^2$ is the discriminant. If $D\leqslant 0$, the eigenvalues are real,

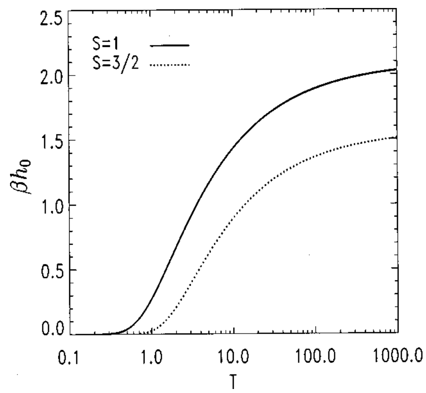

FIG. 1. The critical lines of 1D Ising ferromagnets. The unit of T is k/J.

$$\lambda_1 = 2\sqrt{-Q}\cos(\theta/3) - a_1/3, \tag{33}$$

$$\lambda_2 = 2\sqrt{-Q}\cos(2\pi/3 + \theta/3) - a_1/3, \tag{34}$$

$$\lambda_3 = 2\sqrt{-Q}\cos(4\pi/3 + \theta/3) - a_1/3, \tag{35}$$

where $\cos\theta = R/\sqrt{-Q^3}$. The general condition of phase transition, Eqs. (1) and (8), gives the Yang-Lee edge singularity,

$$\lambda_m^3 + a_1\lambda_m^2 + a_2\lambda_m + a_3 = 0, \tag{36}$$

$$3\lambda_m^2 + 2a_1\lambda_m + a_2 = 0. \tag{37}$$

Equation (37) gives

$$\lambda_m = \sqrt{-Q} - a_1/3. \tag{38}$$

Matching Eq. (38) with, say, Eq. (33), gives $\cos(\theta/3) = 1/2$. Thus $\theta = \pi$ and $\lambda_m = \lambda_1 = \lambda_3$. We see that the Yang-Lee edge singularity does correspond to the twofold degeneracy of the largest eigenvalue. The condition of the Yang-Lee edge singularity becomes $\cos\theta = -1 = R/\sqrt{-Q^3}$. Substituting the expressions R and Q into $R = -\sqrt{-Q^3}$ gives

$$a_2^3/27 - a_1^2 a_2^2/108 + a_3^2/4 - a_1 a_2 a_3/6 + a_3 a_1^3/27 = 0 \quad (R < 0), \tag{39}$$

where a_1, a_2, and a_3 are given by Eqs. (30)–(32) with $h = ih_0$.

Near the critical line ($\theta = \pi$), the largest and next largest eigenvalues are given by

$$\lambda = \sqrt{-Q} - a_1/3 \pm (\sqrt{3}/3Q)\sqrt{-D}. \tag{40}$$

We see that the nature of singularity is indeed the same as in the case of $S = 1/2$. Therefore critical exponents are the same in both cases. The critical line is plotted in Fig. 1.

C. $S = 3/2$

The transfer matrix is given by

$$V = \begin{pmatrix} c^9 d^3 & c^3 d^2 & c^{-3} d & c^{-9} \\ c^3 d^2 & cd & c^{-1} & c^{-3} d^{-1} \\ c^{-3} d & c^{-1} & cd^{-1} & c^3 d^{-2} \\ c^{-9} & c^{-3} d^{-1} & c^3 d^{-2} & c^9 d^{-3} \end{pmatrix},$$

where $c = e^{\beta J/4}$ and $d = e^{\beta h/2}$. The eigenvalues are obtained from

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{41}$$

where

$$a_1 = -c^9(d^3 + d^{-3}) - c(d + d^{-1}), \tag{42}$$

$$a_2 = (c^{10} - c^6)(d^4 + d^{-4}) + (c^{10} - c^{-6})(d^2 + d^{-2}) + (c^2 - c^{-2}) + (c^{18} - c^{-18}), \tag{43}$$

$$a_3 = (c^{-5} - 2c^{-1} + 2c^7 - c^{11})(d^3 + d^{-3}) + (c^{-17} - 2c^{-9} + c^3 + c^{15} - c^{19})(d + d^{-1}), \tag{44}$$

$$a_4 = (1-c)^6(1+c)^6(1+c^2)^6(1-c+c^2) \times (1+c+c^2)(1+c^4)^2(1-c^2+c^4)c^{-20}. \tag{45}$$

We consider the case in which h is purely imaginary. All of a_1, a_2, a_3, and a_4 are real. Let y_1 be a real root of the cubic equation,

$$y^3 - a_2 y^2 + (a_1 a_3 - 4a_4)y + (4a_2 a_4 - a_3^2 - a_1^2 a_4) = 0. \tag{46}$$

Then the eigenvalues are given by the four roots of the equation, $\lambda^2 + A_\pm\lambda + B_\pm = 0$, i.e.,

$$\lambda = \frac{1}{2}[-A_\pm + \sqrt{A_\pm^2 - 4B_\pm}] \tag{47}$$

and

$$\lambda = \frac{1}{2}[-A_\pm - \sqrt{A_\pm^2 - 4B_\pm}], \tag{48}$$

where

$$A_\pm = \frac{1}{2}[a_1 \pm \sqrt{a_1^2 - 4a_2 + 4y_1}], \quad B_\pm = \frac{1}{2}[y_1 \mp \sqrt{y_1^2 - 4a_4}]. \tag{49}$$

In the gap, the eigenvalues are real. Since $a_1 < 0$, we identify the largest eigenvalue with

$$\lambda_m = -\frac{1}{4}a_1 + \frac{1}{2}\sqrt{a_1^2 - 4a_2 + 4y_1} + \frac{1}{2}\sqrt{A_-^2 - 4B_-}. \tag{50}$$

The condition of phase transition requires that $a_1^2 - 4a_2 + 4y_1 = 0$ or $A_-^2 - 4B_- = 0$. However, $a_1^2 - 4a_2 + 4y_1 = 0$ does not correspond to the edge singularity. So the edge singularity corresponds to $A_-^2 - 4B_- = 0$, i.e.,

$$[a_1 - \sqrt{a_1^2 - 4a_2 + 4y_1}]^2 = 8[y_1 + \sqrt{y_1^2 - 4a_4}]. \tag{51}$$

We see again that the Yang-Lee edge singularity does correspond to the twofold degeneracy of the largest eigenvalue.

We can start with the general condition of phase transition, Eqs. (1) and (8), which gives Eq. (41) and

$$4\lambda_m^3 + 3a_1\lambda_m^2 + 2a_2\lambda_m + a_3 = 0. \tag{52}$$

We obtain the same result as it should be. The critical line is plotted in Fig. 1.

From Eq. (49), we see that the nature of singularity is indeed the same as in the case of $S=1/2$. Therefore critical exponents are the same in all cases, $S=1/2,1,3/2$.

IV. 1D SPIN-1 BLUME-EMERY-GRIFFITHS MODEL

The spin-1 Ising model with nearest-neighbor exchange interactions, both bilinear and biquadratic, and with a crystal-field interaction was introduced by Blume, Emery, and Griffiths [23] to describe phase separation and superfluid ordering in He^3-He^4 mixtures. The model can be used as a lattice gas model to describe phase transitions in simple and multicomponent fluids [24]. The model is now considered a standard model in the tricritical phenomena [25]. The Hamiltonian is given by

$$H=-\sum_{\langle ij\rangle}\left[JS_iS_j+KS_i^2S_j^2+\tfrac{1}{2}LS_iS_j(S_i+S_j)-\Delta S_i^2+hS_j\right], \tag{53}$$

where J, K, L, and Δ are interaction strengths. For the 1D case, the transfer matrix is given by

$$\langle S_j|V|S_{j+1}\rangle=\exp\left[\beta JS_jS_{j+1}+\beta KS_j^2S_{j+1}^2+\tfrac{1}{2}\beta LS_jS_{j+1}(S_j+S_{j+1})-\tfrac{1}{2}\beta\Delta(S_j^2+S_{j+1}^2)+\tfrac{1}{2}\beta h(S_j+S_{j+1})\right], \tag{54}$$

or explicitly

$$V=\begin{pmatrix} e^{\beta(J+K-\Delta+L+h)} & e^{\beta(-\Delta+h)/2} & e^{\beta(-J+K-\Delta)} \\ e^{\beta(-\Delta+h)/2} & 1 & e^{-\beta(\Delta+h)/2} \\ e^{\beta(-J+K-\Delta)} & e^{-\beta(\Delta+h)/2} & e^{\beta(J+K-\Delta-L-h)} \end{pmatrix}.$$

The eigenvalues of the matrix are obtained from

$$\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0, \tag{55}$$

where

$$a_1=-[1+e^{\beta(J+K-\Delta)}(e^{\beta(-h-L)}+e^{\beta(h+L)})], \tag{56}$$

$$a_2=-e^{-\beta\Delta}(e^{\beta h}+e^{-\beta h})+e^{\beta(J+K-\Delta)}(e^{\beta(L+h)}+e^{-\beta(L+h)})+e^{2\beta(J+K-\Delta)}-e^{-2\beta(J-K+\Delta)}, \tag{57}$$

$$a_3=e^{-2\beta(J-K+\Delta)}-e^{2\beta(J+K-\Delta)}-2e^{-\beta(J-K+2\Delta)}+e^{\beta(J+K-2\Delta)}(e^{\beta L}+e^{-\beta L}). \tag{58}$$

Following the same analyses as in the preceding section, we find that the edge singularity is still given by Eq. (39). The nature of singularity remains the same as in the previous cases.

If $\theta=0$ or 2π, i.e., $R=\sqrt{-Q^3}$, then the eigenvalues are $\lambda_m=\lambda_1=2\sqrt{-Q}-a_1/3$ and $\lambda_2=\lambda_3=-\sqrt{-Q}-a_1/3$. This means that the nonlargest eigenvalues become twofold degenerate. It is easy to show that, near $\theta=0$ or 2π, no singularity appears. Therefore, this is a *pseudosingularity*.

The ferromagnetic condition requires that ($T=0$, $h=0$) be the unique critical point when h is real. On the other hand, in order to guarantee the existence of edge singularity, the solution of Eq. (39) for h must be purely imaginary. Also in the gap the eigenvalues must be real. These considerations give $L=0$ and $J>0$, under which the Yang-Lee circle theo-

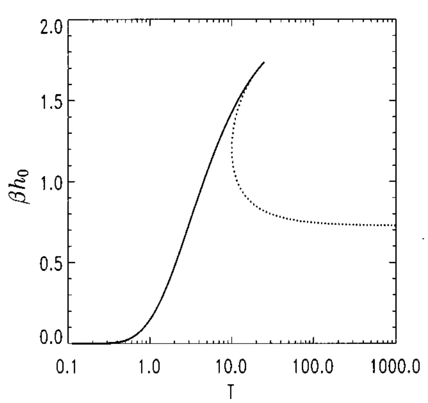

FIG. 2. The type A critical line of a 1D BEG ferromagnet. The unit of T is k/J. $L=0$, $K=1.5J$, and $\Delta=0.5J$. The solid line is the critical line and the dotted line is the pseudocritical line.

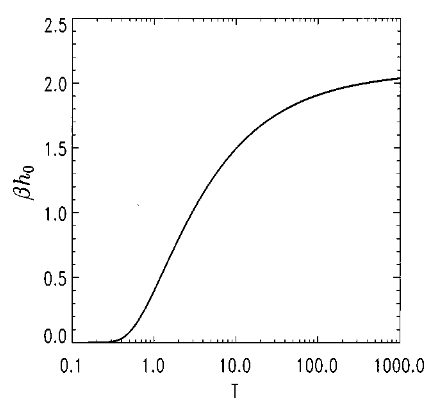

FIG. 3. The type B critical line of a 1D BEG ferromagnet. The unit of T is k/J. $L=0$, $K=0.5J$, and $\Delta=1.25J$.

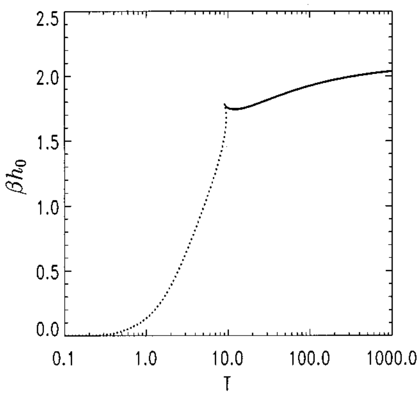

FIG. 4. The type C critical line of a 1D BEG ferromagnet. The unit of T is k/J. $L=0$, $K=0.5J$, and $\Delta=4J$. The solid line is the critical line and the dotted line is the pseudocritical line.

rem is valid. We now discuss several cases.

A. $J+K-\Delta>0$

If $J+K>0$, as $T\to 0$ and $h\to 0$, $a_1\to -2e^{\beta(J+K-\Delta)}$, $a_2\to e^{2\beta(J+K-\Delta)}$, and $a_3\to -e^{2\beta(J+K-\Delta)}$. So $R\to(-1/18)e^{3\beta(J+K-\Delta)}<0$ and $R=-\sqrt{-Q^3}$. $(T=0, h=0)$ is a critical point. We have calculated critical lines with different sets of (K,Δ). We found the following phenomena. (i) If there exists a point (T_t, ih_t) satisfying $Q=R=0$, i.e., $3a_2-a_1^2=27a_3-a_1^3=0$, then the critical line exists between $0\leq T_c\leq T_t$. The end point is the tricritical point (T_t, ih_t). The tricritical point corresponds to the triple degeneracy of the eigenvalues. Near the tricritical point, the nature of singularity is identical to that of the edge singularity. Therefore the tricritical exponents are all the same. The pseudocritical line of the edge pseudosingularity emerges also from the tricritical point, as shown in Fig. 2. The physical picture is that in the gap the eigenvalues are real and no Yang-Lee zeros appear. At the edge, the nonlargest eigenvalues become twofold degenerate precisely. On the rest of the imaginary magnetic field axis, the eigenvalues become complex and Yang-Lee zeros appear. We call this kind of critical line type A. (ii) If the tricritical point does not exist, the shape of the critical line is similar to that of a pure Ising ferromagnet with $S=1$. As $T_c\to\infty$, $\beta_c h_0$ approaches $2\pi/3$, as shown in Fig. 3. We call this type of critical line type B.

If $J+K<0$, the solution of Eq. (39) for h is not purely imaginary and the Yang-Lee circle theorem is not valid.

B. $J+K-\Delta<0$

If $\Delta>0$, as $T\to 0$ and $h\to 0$, $a_1\to -1$, $a_2\to 0$, and $a_3\to 0$. So $R\to 1/27>0$ and $(T=0, h=0)$ is not a critical point. We have tried several different sets of (K,Δ). We found the following phenomena. (i) For some combinations of J, K, and Δ, the critical line begins from a tricritical point and ends at infinity. The rest is the pseudocritical line, beginning from $T=0$ and $h=0$ and ending at the tricritical point, as shown in Fig. 4. We call this kind of critical line type C. (ii) For other combinations of J, K, and Δ, no Yang-Lee edge singularity exists. However, the edge pseudosingularity exists. To our surprise, the Yang-Lee circle theorem is valid, but no edge singularity exists. The pseudocritical line is shown in Fig. 5. We call this kind of pseudocritical line type D.

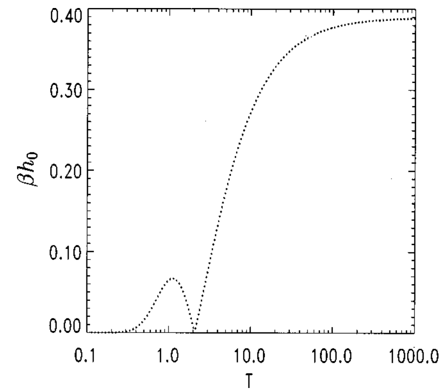

FIG. 5. The type D pseudocritical line of a 1D BEG ferromagnet. The unit of T is k/J. $L=0$, $K=2J$, and $\Delta=4J$.

If $\Delta<0$, the solution of Eq. (39) for h is not purely imaginary and the Yang-Lee circle theorem is not valid.

C. $J+K-\Delta=0$

If $\Delta>0$, as $T\to 0$ and $h\to 0$, $a_1\to -3$, $a_2\to 3$, and $a_3\to -1$, and $R\to 0$ and $Q\to 0$. This is a marginal case. We have calculated several cases. It is found that either the edge pseudosingularity (type D) or the edge singularity (type B) exists, depending on the combinations of J, K, and Δ, as shown in Figs. 3 and 5. If $\Delta<0$, the solution of Eq. (39) for h is not purely imaginary and the Yang-Lee circle theorem is not valid.

V. CONCLUSION

We have shown that phase transition is marked by the occurrence of the twofold degeneracy of the largest eigenvalue of the transfer matrix. For a 1D lattice system in a real magnetic field, if the interaction is finite range, no phase transition occurs at a finite temperature. For the same system in a purely imaginary magnetic field, the nature of the Yang-Lee edge singularity is universal, independent of spin and interaction strengths. The critical exponents satisfy all scaling relations. The edge singularity corresponds to the twofold degeneracy of the largest eigenvalues of the transfer matrix. For a one-dimensional spin-1 Blume-Emery-Griffiths model with $L=0$ and $J>0$, the Yang-Lee circle theorem is valid in general, except for some special cases. For some combinations of J, K, and Δ, the tricritical phenomenon and the edge pseudosingularity exist. The edge pseudosingularity corresponds to the twofold degeneracy of the nonlargest eigenvalues. The tricritical point corresponds to the triple degenerate eigenvalues. The tricritical exponents are the same as those of the edge singularity. For some combinations of J, K, and Δ, only the edge pseudosingularity exists, but the Yang-Lee circle theorem is still valid. All one-dimensional Ising models with arbitrary spin, including higher-order and finite-range interactions, belong to the same universality class of the edge singularity.

ACKNOWLEDGMENT

This work was funded by Pohang University of Science and Technology.

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